Portfolio Theory

- We develop powerful models and theories about the right way to make investment and financing decisions. We argue that all of these conclusions are conditional on the acceptance of value maximization as the only objective in decision-making.

- We have to choose the right objective: An objective specifies what a decision maker is trying to accomplish and by so doing provides measures that can be used to choose between alternatives.

- In most firms, the managers of the firm, make the decisions about where to invest or how to raise funds for an investment.

- In most cases, the objective is stated in terms of maximizing some function or variable, such as profits or growth or minimizing some function or variable, such as risk or costs.
Objectives when making decision can be stated broadly as maximizing the value of the entire business, more narrowly as maximizing the value of the equity state in the business or even more narrowly as maximizing the stock price for a publicly traded firm.

If the objective when making decisions is to maximize firm value, there is a possibility that what is good for the firm may not be good for society. In addition when managers acts as agents for the owners (stockholders), there is the potential for a conflict of interest between stockholder and managerial interest.

When the objective is narrowed further to one of maximizing stock price, inefficiencies in the financial market may lead to misallocation of resources and to bad decisions.
Why corporate finance focuses on Stock Price Maximization

- Stock prices are the most observable of all measures that can be used to judge the performance of a publicly traded firm.
- Stock prices are updated constantly to reflect new information coming out about the firm. Thus, managers receive instantaneous feedback from investors on every action that they take.
- If investors are rational and markets are efficient, stock prices will reflect the long-term effects of decisions made by the firm.
- Choosing stock price maximization as an objective allows us to make categorical statement about the best way to pick projects and finance them and to test these statement with empirical observation.
Are Markets short-term?

- There are many who believe that stock price maximization leads to a short-term focus for managers. They reason that Stock prices are determined by traders and short-term investors who hold stocks for short periods and spend their time trying to forecast next quarter’s earnings but most of the empirical evidence have suggested that markets are much more for long-term.

- There are hundreds small firms, that do not have any current earnings and cash flows and do not expect to have any in the near future but are still able to raise substantial amounts of money on the basis of expectations of success in the future.

- Evidence suggests that markets do value future earnings and cash flows too much. Particularly, stocks with low price-earnings ratios and high current earnings are generally underpriced relative to stocks with high price-earnings ratios.
Alternatives to stock price maximization

- Maximize market share
  - In the 1980s, Japanese firms focused their attention on increasing market share. Proponents of this objective note that market share is observable and measurable like market price and does not require any of the assumptions about efficient financial markets that are needed to justify the stock price maximization objective.
  - Underlying the market share maximization objective is the belief that higher market share will mean more pricing power and higher profits in the long run. However, if higher market share does not yield higher pricing power, and the increase in market share is accompanied by lower or even negative earnings, firms that concentrate on increasing market share can be worse off as a consequence.
Maximize Profit

These are objectives that focus on profitability rather than values. The rationale is that profit can be measured more easily than value, and that higher profits translate into higher value in the long run.

There are at least two problems with these objectives. First, the emphasis on current profitability may result in short-term decisions that maximize profits now at the expenses of long-term profits and value. Second, the notion that profits can be measured more precisely than value may be incorrect, given the leeway that accountants have to shift profits across periods.

Remark

Therefore given the limitations of the alternatives, we believe that managers should make decisions that increase the long-term value of the firm and then try to provide as much information as they can about the consequences of these decisions to financial markets.
Motivating example

- Consider the simplest of all worlds, a one-person/one-good economy with no uncertainty. The decision maker, Robinson, must choose between consumption now and consumption in the future (Investment).
- In order to decide, he needs two types of informations:
  - He needs to understand his own subjective trade-offs between consumption now and consumption in the future (this information is embodied in the utility and indifference curves).
  - He must know the feasible trade-offs between present and future consumptions that are technologically possible.
Fisher Separation theorem

Theorem

Given perfect and complete capital markets, the production decision is governed solely by an objective market criterion (represented by maximizing attained wealth) without regard to individuals’ subjective preference that enter into their consumption decisions.

In other words, the Separation theorem says that investment decisions and financing decisions should be made independent of one another. This proposition was identified by Irving Fisher in the 1930s and was formally set out by Hirshleifer (1958).

Remark

(Implications for corporate policy)

- An important implication for corporate policy is that the investment decision can be delegated to the managers. Given the same opportunity set, every investor will make the same production decision regardless of the shape of his or her indifference curves.
Utility theory given uncertainty

We recall that:

- Utility is defined as the satisfaction that an individual obtains from a particular course of action, such as the consumption of a good.
- The notion of utility provides a means of expressing individual tastes and preferences.
- Utility and differing levels of it are frequently represented graphically by indifference curves, each one showing a constant level of utility or satisfaction for differing combinations of related factors.
Utility Function

- Just as we always draw indifference curves with a particular shape (i.e. downward-sloping and convex to the origin), so we usually draw utility function with a particular shape.
- We would like to use utility function to allow for the assignment of unit measure (a number) to various alternatives to help make a choice.
- Utility function have two properties:
  - Order preserving:
    - If we measure the utility of $x$ as greater than the utility of $y$, $U(x) > U(y)$ then $x$ is actually preferred to $y$ ($x > y$).
  - Expected utility can be used to rank combinations of risky alternatives:
    - $U[G(x, y : \alpha)] = \alpha U(x) + (1 - \alpha)U(y)$
Axioms of choice under uncertainty

(Von Neumann and Morgenstern’s axioms)

The expected utility theorem can be derived formally from the following four axioms. In other words, an investor whose behavior is consistent with these axioms will always make decisions in accordance with the expected utility theorem.

1. Comparability (Completeness):
   - For an entire set, S, of uncertain alternatives, an individual can say either that outcome \( x \) is preferred to outcome \( y \) (\( x > y \)) or \( y \) is preferred to \( x \) (\( y > x \)) or the individual is indifferent as to \( x \) and \( y \) (\( x \sim y \)).

2. Transitivity (Consistency):
   - If an individual prefers \( x \) to \( y \) and \( y \) to \( z \), then \( x \) is preferred to \( z \). That is if \( x > y \) and \( y > z \) then \( x > z \). Similarly, if \( x \sim y \) and \( y \sim z \) then \( x \sim z \). This implies that investors are consistent in their rankings of outcomes.
3. (Strong) independence:

- Suppose we construct a gamble where an individual has a probability \( \alpha \) of receiving outcome \( x \) and a probability of \( (1 - \alpha) \) of receiving outcome \( z \). (We write \( G(x, z : \alpha) \)).

- Strong independence says that if the individual is indifferent as to \( x \) and \( y \), then he will also be indifferent as to a first gamble, set up between \( x \) with probability \( \alpha \) and mutually exclusive outcome, \( z \) and a second gamble, set up between \( y \) with probability \( \alpha \) and the same mutually exclusive outcome, \( z \).

- If \( x \sim y \) then \( G(x, z : \alpha) \sim G(y, z : \alpha) \).
4. Measurability (Certainty equivalence):

- If outcome $y$ is preferred less than $x$ but more than $z$, then there is a unique $\alpha$ (probability) such that the individual will be indifferent between $y$ and a gamble between $x$ with probability $\alpha$ and $z$ with probability $(1 - \alpha)$.
- If $x > y \geq z$ then there exist a unique $\alpha$ such that $y \sim G(x, z : \alpha)$.
- It represents the certain outcomes or level of wealth that yields the same certain utility as the expected utility yielded by the gamble or lottery involving outcomes $x$ and $z$. $y$ can also be interpreted as the maximum price that an investor would be willing to pay to accept a gamble.
5. Ranking:

- If alternative y and u both lie somewhere between x and z and we can establish a gamble such that an individual is indifferent between y and a gamble between x (with probability $\alpha_1$) and z, while also indifferent between u and a second gamble, this time between x (with probability $\alpha_2$) and z, then if $\alpha_1$ is greater than $\alpha_2$, y is preferred to u.

- If $x \geq y \geq z$ and $x \geq u \geq z$ then if $y \sim G(x, z : \alpha_1)$ and $u \sim G(x, z : \alpha_2)$ it follows that if $\alpha_1 > \alpha_2$ then $y > u$ or if $\alpha_1 = \alpha_2$ then $y \sim u$. 
**Example**

Suppose that an investor is asked to choose between various pairs of strategies and responds as follows:

<table>
<thead>
<tr>
<th>Choose between:</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>B and D</td>
<td>B</td>
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<tr>
<td>A and D</td>
<td>D</td>
</tr>
<tr>
<td>C and D</td>
<td>indifferent</td>
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<tr>
<td>B and E</td>
<td>B</td>
</tr>
<tr>
<td>A and C</td>
<td>C</td>
</tr>
<tr>
<td>D and E</td>
<td>indifferent</td>
</tr>
</tbody>
</table>

Assuming that the investor’s preferences satisfy the four axioms discussed above, how does he rank the five investments A to E?
Solution

From the response we can note immediately that:

\[ B > D, \quad D > A, \quad C = D, \quad B > E, \quad C > A, \quad D = E \]

Hence, transitivity then implies that:

\[ B > D > A \]
\[ C = D = E \]

And so we have that:

\[ B > C = D = E > A \]
Example

Suppose we arbitrary assign a utility of -10 utiles to a loss of $1,000 and ask the following question: When we are faced with a gamble with probability $\alpha$ of winning $1,000 and probability $(1 - \alpha)$ of losing $1,000, what probability would make us indifferent between the gamble and $0 with certainty?
Solution

- Mathematically:

\[
0 \sim G(1,000, -1000 : \alpha)
\]

\[
U(0) = \alpha U(1,000) + (1 - \alpha) U(-1,000)
\]

- Suppose that the probability of winning $1,000 must be 0.6 in order for us to be indifferent between the gamble and a sum $0. By assuming that the utility of $0 with certainty is zero and substituting \(U(-1,000) = -10\) and \(\alpha = .6\) into the above equation, the utility of $1,000:

\[
U(1000) = \frac{(1 - \alpha)U(-1,000)}{\alpha} = \frac{(1 - .6)(-10)}{.6} = 6.7 \text{ utiles}
\]
Finding expected utility

- We assume that an investor has a utility function $U(W)$, which attaches a numerical value to the satisfaction attained from a level of wealth $W$, at some future date - for example, the next period.
- Decisions are made on the basis of maximizing the expected value of utility under the investor’s particular belief about the probability of different outcomes.
- If we consider a risky asset as a lottery (gamble) with a set of $N$ possible outcomes $(W_1, \cdots, W_N)$, each with associated probabilities of occurring $(p_1, \cdots, p_N)$, then the expected utility yielded by investment in this risky asset is given by:

$$E[U(W)] = \sum_i p_i U(W_i)$$
Remark

- Given the five axioms of rational investor behavior and the additional assumption that all investors always prefer more wealth to less, we can say that investors will always seek to maximize their expected utility of wealth.
- In other words, they will seem to calculate the expected utility of wealth for all possible alternative choices and then choose the outcomes that maximizes their expected utility of wealth.

Theorem

The expected utility theorem says that when making a choice an individual should choose the course of action that yields the highest expected utility—and not the course of action that yields the highest expected wealth.
Variations in beliefs

- Each investor may have different beliefs about both:
  - The values of the outcomes \( (W_1, \ldots, W_N) \), and also
  - The values of the associated probabilities \( (p_1, \ldots, p_N) \)

  i.e. the characteristics—expected return and variance of returns—offered by each risky asset.

- Each investor will also have different preferences as regards the trade-off between risk (or variability of return) and expected returns, which will be reflected in the characteristics of the utility function that he uses to make his investment decisions.

- By combining his beliefs about the set of available assets with his utility function, he can determine the optimal investment portfolio in which to invest, i.e., that which maximizes his expected utility in that period.
Risk attitudes

In general, if the utility of expected wealth is greater than the expected utility of wealth, the individual will be risk averse. The three definitions are

1. If \[ U[E(W)] > E[U(W)] \], then we have risk aversion. His utility function condition is \[ U''(W) < 0 \] i.e for a risk-averse, utility is a (strictly) concave function of wealth. A risk-averse person dislikes risk and will always reject a fair gamble.

2. If \[ U[E(W)] = E[U(W)] \], then we have risk neutrality. His utility function condition is \[ U''(W) = 0 \]. A risk-neutral person is indifferent to risk and hence between accepting or rejecting a fair gamble, which offers no expected gain.

3. If \[ U[E(W)] < E[U(W)] \], then we have risk loving. His utility function condition is \[ U''(W) > 0 \]. A risk-loving person likes risk and will always accept a fair gamble.
Example

Investor A has an initial wealth of $100 and a utility function of the form: \( U(W) = \log(W) \) where \( W \) is her wealth at any time. Investment \( Z \) offers her a return of \(-18\%\) or \(+20\%\) with equal probability.

(i) What is her expected utility if she invests nothing in Investment \( Z \)?

(ii) What is her expected utility if she invests entirely in Investment \( Z \)?

(iii) What proportion \( a \) of her wealth should she invest in Investment \( Z \) to maximize her expected utility? What is her expected utility if she invests this proportion in Investment \( Z \)?
Solution

(i) The expected utility of Investor A is \( = \log(100) = 4.605 \)

(ii) The expected utility of Investor A is:
\[
= 0.5 \times \log(0.82 \times 100) + 0.5 \times \log(1.2 \times 100) = 4.597
\]

(iii) The expected utility of Investor A is given by:
\[
E[U(W)] = \sum_{i=1}^{n} p_i U(W_i)
\]
\[
= 0.5\{\log[(1 - 0.18a)100]\} + 0.5\{\log[(1 + 0.2a)100]\}
\]
\[
= 0.5\{\log[100 - 18a]\} + 0.5\{\log[100 + 20a]\}
\]

We differentiate with respect to \( a \) to find a maximum
\[
\frac{dE[U(W)]}{da} = 0.5 \times \frac{-18}{100 - 18a} + 0.5 \times \frac{20}{100 + 20a}
\]
\[
= \frac{-9}{100 - 18a} + \frac{10}{100 + 20a}
\]
We then set equal to zero
\[
\frac{9}{100 - 18a} = \frac{10}{100 + 20a}
\]

Solving, we find \( a = 0.2777 \).

Checking to see if this gives a maximum:
\[
\frac{d^2 E[U(W)]}{da^2} = \frac{+9(-18)}{(100 - 18a)^2} + \frac{-10(20)}{(100 + 20a)^2}
\]

This gives a negative value so it is a maximum.

Finding the expected utility from investing 27.77% in Investment \( Z \):
\[
E[U(W)] = 0.5\log[(1 - 0.18(0.2777))100] + 0.5\log[(1 + 0.2(0.2777))100]
\]
\[
= 0.5\log[100 - 18(0.2777)] + 0.5\log[100 + 20(0.2777)]
\]
\[
= 4.6065
\]
Some commonly used utility functions

- The quadratic utility function: \( U(W) = a + bW + cW^2 \) which can as well be written as \( U(W) = W + dW^2 \) since adding a constant will not affect the decision making

- The log utility function: \( U(W) = \ln(W), \quad (W > 0) \)

- The power utility function: \( U(W) = \frac{W^{\gamma} - 1}{\gamma}, \quad (W > 0) \)
Example

You have a logarithm utility function $U(W) = \ln W$ and your current level of wealth is $5,000$

(a) Suppose you are exposed to a situation that results in a 50/50 chance of winning or losing $1,000$. If you can buy insurance that completely removes the risk for a fee of $125$, will you buy it or take the gamble?

(b) Suppose you accept the gamble outlined in (a) and lose, so that your wealth is reduced to $4000$. If you are faced with the same gamble and have the same offer of insurance as before will you buy the insurance the second time round?
Solution

(a)

\[ E[U(W)] = 0.5 \ln(4,000) + 0.5 \ln(6,000) \]
\[ = 0.5(8.29415) + 0.5(8.699515) = 8.4967825 \]
\[ e^{\ln W} = W \implies e^{8.496782} = 4,898.89 = W \]

Therefore, the individual would be indifferent between the gamble and $4,898.98 for sure. This amount to a risk premium of $101.02. Therefore, he would not buy insurance for $125.
(b) The second gamble, given his first loss, is $4,000 plus or minus $1,000. Its expected utility is

\[
E[U(W)] = 0.5 \ln(3,000) + 0.5 \ln(5,000)
\]

\[
= 0.5(8.006368) + 0.5(8.517193) = 8.26178
\]

\[
e^{\ln W} = e^{8.26178} = $31872.98 = W
\]

Now the individual would be willing to pay up to $127.02 for insurance since insurance cost only $125, he will buy it.
Questions

Consider the following utility function: \( U(W) = -e^{-aW}, \quad a > 0 \) Derive expressions for the absolute risk aversion and relative risk aversion measures. What does the latter indicate about the investor’s desire to hold risky asset?

Solution

- The utility function \( U(W) = -e^{-aW} \) is such that:
  \[ U'(W) = ae^{-aW} \quad \text{and} \quad U''(W) = -a^2 e^{-aW}. \]

  Thus:
  \[ A(W) = \frac{-U''(W)}{U'(W)} = a > 0 \quad \text{and} \quad R(W) = \frac{-WU''(W)}{U'(W)} = aW > 0. \]

  Hence, as the absolute risk aversion is constant and independent of wealth the investor must hold the same absolute amount of wealth in risky assets. Both this, and the fact that the relative risk aversion increases with wealth, are consistent with a decreasing proportion of wealth being held in risky assets as wealth increases.
Limitations of utility theory

The expected utility theorem is a very useful device for helping to condition our thinking about decisions, because it focuses attention on the types of tradeoffs that have to be made. However, the expected utility theorem has several limitations that reduce its relevance for risk management purpose:

- To calculate expected utility, we need to know the precise form and shape of the individual’s utility function. Typically, we do not have such information. Usually, the best we can hope for is to identify a general feature, such as risk aversion, and to use the rule to identify broad types of choices that might be appropriate.
For corporate risk management, it may not be possible to consider a utility function for the firm as though the firm was an individual. The firm is a coalition of interest groups, each having claims on the firm. The decision process must reflect the mechanisms with which these claims are resolved and how this resolution affects the value of the firm. Furthermore, the risk management costs facing a firm may be only one of a number of risky projects affecting the firm’s owners (and other claimholders). The expected utility theorem is not an efficient mechanism for modeling the interdependence of these sources of risk.

Alternative decision rules that can be used for risky choices include the mean-variance rule and stochastic dominance.
## Investment risk

### Remark

Conduct some brief research about investment risks.

A simple question could be:

### Questions

State five possible types of risk that might be relevant in an investment context?
Solution

1. Default or credit risk - the other party to an investment deal fails to fulfil their obligations.

2. Inflation risk - inflation is higher than anticipated, so reducing real returns.

3. Exchange rate or currency risk - exchange rate moves in an unanticipated way.

4. Reinvestment risk - stems from the uncertainty concerning the terms on which investment income can be reinvested.

5. Marketability risk - the risk that you might be unable to realise the true value of an investment if it is difficult to find a buyer.
Introduction

- Risk, in traditional terms, is viewed as a negative and something to be avoided.
- In finance, risk can be viewed as the trade off that every investor and business has to make - between the "higher rewards" that potentially come with the opportunity and the "higher risk" that has to be borne as a consequence of the danger.
- The key test in finance is to ensure that when an investor is exposed to risk that he or she is "appropriately" rewarded for taking this risk.
- In our study we lay the foundations for analyzing risk in finance and present alternative models for measuring risk and converting these risk measures into "acceptable" hurdle rates.
Motivation and Perspective in Analyzing Risk

- A good model for risk and return provides us with the tools to measure the risk in any investment and uses that risk measure to come up with appropriate expected return on that investment; this expected return provides us with the hurdle rate in project analysis.

- We will argue that risk in an equity investment has to be perceived through the eyes of investors in the firm.

- We will assert that risk has to be measured from the perspective of not just any investor in the stock, but of the marginal investor, defined to be investor most likely to be trading on the stock at any given point in time.

  - The objective in corporate finance is the maximization of firm value and stock price. If we want to stay true to this objective, we have to consider the viewpoint of those who set the stock prices and they are the marginal investors.
Measures of risk

- In finance it is often assumed that the key factors influencing investment decisions are "risk" and "return".
- Most mathematical investment theories of investment risk use variance of return as the measure of risk. Examples include (mean-variance) portfolio theory and the capital asset pricing model discussed later.
- However, it is not obvious that variance necessarily corresponds to investors’ perception of risk and other measures have been proposed as being more appropriate.
- Some investors might not be concerned with the mean and variance of returns, but simpler things such as the maximum possible loss. Alternatively, some investors might be concerned not only with the mean and variance of returns, but also more generally with other higher moments of returns, such as the skewness of returns.

- For example, although two risky assets might yield the same expectation and variance of future returns, if the returns on Asset A are positively skewed, whilst those on Asset B are symmetrical about the mean, then Asset A might be preferred to Asset B by some investors.

- In addition to the expected return, an investor now has to consider the spread of the actual returns around the expected return which is captured by the variance or standard deviation of the distribution; the greater the deviation of the actual returns from expected returns, the greater the variance.

- The bias towards positive or negative returns is captured by the skewness of the distribution.
The shape of the tails of the distribution is measured by the kurtosis of the distribution; fatter tails lead to higher kurtosis. In investment terms, this captures the tendency of the price of this investment to “jump” in either direction.

In the special case of the normal distribution, returns are symmetric and investors do not have to worry about skewness and kurtosis, since there is no skewness and normal distribution is defined to have a kurtosis of zero.

In this case, investment can be measured on only two dimensions: the ‘expected return’ on the investment comprises the reward and the variance in anticipated returns comprises the risk on the investment.
Variance of return

- For a continuous distribution, variance of return is defined as:

\[ \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx \]

Where \( \mu \) is the mean return at the end of the chosen period and \( f(x) \) is the probability density function of the return.

"Return" here means the proportionate increase in the market value of the asset.

The units of variance are "\%\%", which means "per 100 per 100" e.g.

\[(4\%)^2 = 16\%\% = 0.16\% = 0.0016\]

- For a discrete distribution, variance of return is defined as:

\[ \sum_{x} (x - \mu)^2 P(X = x) \]

where \( \mu \) is the mean return at the end of the chosen period.

- Variance in Return is a measure of the squared difference between the actual returns and the expected returns on an investment.
Example

1 Investment return (% pa), $X$, on a particular asset are modelled using a probability distribution with density function:

$$f(x) = 0.00075(100 - (x - 5)^2), \quad \text{where} \quad -5 \leq x \leq 15$$

Calculate the mean return and the variance of return.

2 Investment return (% pa), $X$, on a particular asset are modelled using the probability distribution:

<table>
<thead>
<tr>
<th>$X$</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>-7</td>
<td>0.04</td>
</tr>
<tr>
<td>5.5</td>
<td>0.96</td>
</tr>
</tbody>
</table>

Calculate the mean return and variance of return.
Solution

(1)
The density function is symmetrical about $x = 5$. Hence the mean return is 5%. Alternatively, this could be found by integrating as follows:

$$E[X] = 0.00075 \int_{-5}^{15} x(100 - (x - 5)^2)dx = 0.00075 \int_{-5}^{15} (75x + 10x^2 - x^3)dx$$

$$= 0.00075 \left[ \frac{75}{2}x^2 + \frac{10}{3}x^3 - \frac{1}{4}x^4 \right]_{-5}^{15} = 0.00075[7031.25 - 364.5833] = 5 \text{ ie } 5\%$$

The variance is given by:

$$\text{var}[X] = 0.00075 \int_{-5}^{15} (5 - x)^2(100 - (x - 5)^2)dx = 0.00075 \int_{-5}^{15} 100(x - 5)^2 - (x - 5)^4dx$$

$$= 0.00075 \left[ \frac{100}{3}(x - 5)^3 - \frac{1}{5}(x - 5)^5 \right]_{-5}^{15} = 0.00075[13,333.33 - (-13,333,33)] = 20 \text{ ie } 20\%pa$$

Alternatively, you may have calculated the variance using the formula: $\text{var}[X] = E[X^2] - E[X]^2$, where $E[X^2]$ can be found by integration to be 45%. 
(2)
The mean return is given by:

\[ E[X] = -7 \times 0.04 + 5.5 \times 0.96 = 5 \text{ ie } 5\% \text{ pa} \]

The variance of return is given by:

\[ \text{var}[X] = (5 - (-7))^2 \times 0.04 + (5 - 5.5)^2 \times 0.96 = 6 \text{ ie } 6\%\% \text{ pa} \]

Alternatively, you may have calculated the variance using the formula:

\[ \text{var}[X] = E[X^2] - E[X]^2, \]

where \( E[X^2] \) is 31\%. 
Variance has the advantage over most other measure in that it is mathematically tractable and the mean-variance leads to elegant solutions for optimal portfolios. This ease of use should not be lightly disregarded, in fact the use of mean-variance theory has been shown to give a good approximation to several other proposed methodologies.

The mean-variance portfolio theory assumes that investors base their investment decisions solely on the mean and variance of investment returns. This assumption is consistent the maximisation of expected utility provided that the investor’s expected utility depends only the mean and variance of investment returns. It can be shown that this is the case if:

- the investor has a quadratic utility function, and/or
- Investment returns follow a distribution that is characterised fully by its first two moments, such as the normal distribution.
The Skewness of a continuous probability distribution is defined as the third central moment:

\[
Skew = \int_{-\infty}^{\infty} (x - \mu)^3 f(x) \, dx
\]

It is a measure of the extent to which a distribution is asymmetric about its mean. For example, the normal distribution is symmetric about its mean and therefore has zero skewness, whereas the lognormal distribution is positively skewed.

The Kurtosis of a continuous probability distribution is defined as the fourth central moment:

\[
K = \int_{-\infty}^{\infty} (x - \mu)^4 f(x) \, dx
\]

It is a measure of the "peakedness" or "pointedness" of a distribution.
Semi-variance of return

The main argument against the use of variance as a measure of risk is that most investors do not dislike uncertainty of return as such; rather they dislike the possibility of low returns.

- For example, all investors would choose a security that offered a chance of either a 10% or 12% return in preference to one that offered a certain 10%, despite the greater uncertainty associated with the former.

One measure that seeks to quantify this view is downside semi-variance (or simply semi-variance). For a continuous random variable, this is defined as:

\[
\int_{-\infty}^{\mu} (\mu - x)^2 f(x) dx
\]
For a discrete random variable, the downside semi-variance is defined as:

\[
\sum_{x<\mu} (\mu - x)^2 P(X = x)
\]

Semi-variance is not easy to handle mathematically and it takes no account of variability above the mean. Furthermore if returns on assets are symmetrically distributed semi-variance is proportional to variance.

Questions

- What is the relationship between semi-variance and variance for the normal distribution?
- Calculate the downside semi-variance of return for the asset modelled in the first and second questions given previously.
Solution

- The normal distribution is symmetrical. Hence the semi-variance is half of the variance.

- The continuous distribution in the Question (1) is symmetrical. Therefore, the downside semi-variance is half the variance, ie 10%.

- For the discrete distribution in Question (2), the downside semi-variance is given by:

\[ \sum_{x<5} (5 - x)^2 P(X = x) = (5 - (-7))^2 \times 0.04 = 5.76 \text{ ie } 5.76\% \text{ pa.} \]
Shortfall probabilities

- A shortfall probability measures the probability of returns falling below a certain level. For continuous variables, the risk measure is given by:
  \[
  \text{Shortfall probability} = \int_{-\infty}^{L} f(x) \, dx \quad \text{where } L \text{ is a chosen benchmark level.}
  \]

- For discrete random variables, the risk measure is given by:
  \[
  \text{Shortfall probability} = \sum_{x < L} P(X = x).
  \]

- The benchmark level can be expressed as the return on a benchmark fund if this is more appropriate than an absolute level. In fact any of the risk measures discussed can be expressed as measures of the risk relative to a suitable benchmark which may be an index, a median fund or some level of inflation.

- \( L \) could alternatively relate to some pre-specified level of surplus or fund solvency.

- The main advantages of shortfall probability are that it is easy to understand and calculate.
Questions

- Calculate the shortfall probability for the asset modelled in Question (1) and (2) where the benchmark return is 0% pa.
- What is the main drawback of the shortfall probability as a measure of investment risk?
Solution

The shortfall probability is given by:

\[
P(X < 0) = 0.00075 \int_{-5}^{0} (100 - (x - 5)^2)dx = 0.00075 \left[100x - \frac{1}{3}(x - 5)^3\right]_{-5}^{0}
= 0.00075 \left[41.6667 - (-166.6667)\right] = 0.15625.
\]

The shortfall probability is given by:

\[
P(X < 0) = 0.04
\]
Disadvantage of using shortfall probability

- The shortfall probability gives no indication of the magnitude of any shortfall (being independent of the extent of any shortfall).
- For example, consider two security that offer the following combinations of returns and associated probabilities:
  - Investment A: 100% with probability of 0.9 and 9.9% with probability of 0.1
  - Investment B: 10.1% with probability of 0.91 and 0% with probability of 0.09
- An investor who chooses between them purely on the basis of the shortfall probability based upon a benchmark return of 10% would choose Investment B!
Value at risk

- Value at Risk (VaR) generalises the likelihood of under-performing by providing a statistical measure of downside risk.
- For a continuous random variable, Value at Risk can be determined as:

\[ \text{VaR}(X) = -t \quad \text{where} \quad P(X < t) = p \]

VaR assesses the potential losses on a portfolio over a given future time period with a given degree of confidence.
- For example, if we adopt a 99% confidence limit, the VaR is the amount of loss that will be exceeded only one time in hundred over a given time period and we would need to find \( t \) such that \( P(X < t) = 0.01 \).
- For a discrete random variable, VaR is defined as:

\[ \text{Var}(X) = -t \quad \text{where} \quad t = \max\{x : P(X < x) \leq p\} \]
Remark

Note that Value at Risk is a "loss amount". Therefore:

- a positive Value at Risk (a negative $t$) indicates a loss
- a negative Value at Risk (a positive $t$) indicates a profit
- Value at Risk should be expressed as a monetary amount and not as a percentage.

The problem is that in practice VaR is usually calculated assuming that investment returns are normally distributed.
Calculate the VaR over one year with a 95% confidence limit for a portfolio consisting of $100m invested in the asset modelled in question (1).

Calculate the 95% VaR over one year with a 95% confidence limit for a portfolio consisting of $100m invested in the asset modelled in Question (2).

Calculate the 97.5% VaR one year for a portfolio consisting of $200m invested in shares. You should assume that the return on the portfolio of shares is normally distributed with mean 8% pa and standard deviation 8% pa.
We start by finding \( t \), where \( P(X < t) = 0.05 \):

\[
\Rightarrow 0.00075 \int_{-5}^{t} 100 - (x - 5)^2 \, dx = 0.05 \quad \Rightarrow \quad 0.00075 \left[ 100x - \frac{1}{3}(x - 5)^3 \right]_{-5}^{t} = 0.05
\]

Since the equation in the brackets is a cubic in \( t \), we are going to need to solve this equation numerically, by trial and error.

\[
t = -3 \quad \Rightarrow \quad 0.00075 \left[ 100x - \frac{1}{3}(x - 5)^3 \right]_{-5}^{-3} = 0.028
\]

and

\[
t = -2 \quad \Rightarrow \quad 0.00075 \left[ 100x - \frac{1}{3}(x - 5)^3 \right]_{-5}^{-2} = 0.06075
\]

interpolating between the two gives

\[
t = -3 + \frac{0.05 - 0.028}{0.06075 - 0.028} = -2.3
\]

In fact, the true value is \( t = -2.293 \). Since \( t \) is a percentage investment return per annum, the 95% value at risk over one year on a \( \$100 \text{m} \) portfolio is \( \$100 \text{m} \times 2.293\% = \$2.293 \text{m} \). This means that, we are 95% certain that we will not lose more than \( \$2.293 \text{m} \) over the next year.
We start by finding $t$, where $t = \max\{x : P(X < x) \leq 0.05\}$. Now $P(X < -7) = 0$ and $P(X < 5.5) = 0.04$. Therefore $t = 5.5$.

Since $t$ is a percentage investment return per annum, the 95% value at risk over one year on a $100 m$ portfolio is $100 m \times -5.5\% = -5.5 m$. This means that, we are 95% certain that will not make profits of less than $5.5 m$ over the next year.

We start by finding $t$, where:

$$P(X < t) = 0.025, \text{ where } X \sim N(8, \sigma^2)$$

Standardising gives:

$$P\left(Z < \frac{t - 8}{\sigma}\right) = \Phi\left(\frac{t - 8}{\sigma}\right) = 0.025$$

But $\Phi(-1.96) = 0.025$, so $t = -7.68$.

Since $t$ is a percentage investment return per annum, the 97.5% value at risk over one year on a $200 m$ portfolio is $200 m \times 7.68\% = 15.36 m$. This means that, we are 97.5% certain that we will not lose more than $15.36 m$ over the next year.
Tail value at risk (TailVar) and expected shortfall

- Closely related to both shortfall probabilities and VaR are the TailVaR and Expected Shortfall measures of risk. The risk measure can be expressed as the shortfall below a certain level.
- For a continuous random variable, the expected shortfall is given by:

\[
\text{Expected shortfall} = E[max(L - X, 0)] = \int_{-\infty}^{L} (L - x)f(x)dx
\]

where \( L \) is the chosen benchmark level.
- For a discrete random variable, the expected shortfall is given by:

\[
\text{Expected shortfall} = E[max(L - X, 0)] = \sum_{x < L} (L - x)P(X = x)
\]

- If \( L \) is chosen to be a particular percentile point on the distribution, then the risk measure is known as the TailVaR.
- The \((1 - p)\) TailVaR is the expected shortfall in the \( p^{th} \) lower tail. So, for the \(99\%)\) confidence limit, it represents the expected loss in excess of the \(1\%)\) lower tail value.
However, Tail VaR can also be expressed as the Expected Shortfall conditional on there being a shortfall. To do this, we would need to take the expected shortfall formula and divide by the shortfall probability.

Downside risk measures have also been proposed based on an increasing function of \((L - x)\), rather than \((L - x)\) itself in the integral above.

In other words, for continuous random variables, we could use a measure of the form:

\[
\int_{-\infty}^{L} g(L - x)f(x)dx
\]

Two particular cases of note are when:

1. \(g(L - r) = (L - r)^2\) — this is the so-called shortfall variance
2. \(g(L - r) = (L - r)\) — the average or expected shortfall measure defined above.

Note also that if \(g(x) = x^2\) and \(L = \mu\), then we have the semi-variance measure defined above.
Example

An investor is contemplating an investment with a return of $R$, where:

\[ R = 300,000 - 500,000U \]

where $U$ is a uniform $[0, 1]$ random variable.

Calculate each of the following four measures of risk:

(a) variance of return

(b) downside semi-variance of return

(c) shortfall probability, where the shortfall level is $100,000$

(d) Value at Risk at the 5% level.
Solution

(a) Variance

$R$ is defined by $R = 300,000 - 500,000U$, where $U$ is $U[0, 1]$. So $R$ has a uniform distribution on the range from $-200,000$ to $300,000$. The variance of $R$ can be calculated directly from the formula $\frac{1}{12}(b - a)^2$:

$$\text{var}(R) = \frac{1}{12}[300,000 - (-200,000)]^2 = \frac{1}{12} \times 500,000^2 = ($144,338$)^2$$

An alternative approach is to evaluate the integral:

$$\int_{-200,000}^{300,000} (\mu - x)^2 f(x)dx$$

where $\mu = \frac{1}{2} (-200,000 + 300,000) = 50,000$ and $f(x) = \frac{1}{500,000}$.

(b) Downside semi-variance

Since the uniform distribution is symmetrical, the semi-variance is just half the "full" variance:

$$\text{semi-variance} = \frac{1}{2} \times \frac{1}{12} \times 500,000^2 = ($102,062$)^2$$

Alternatively, you can evaluate the integral

$$\int_{-200,000}^{50,000} (\mu - x)^2 f(x)dx$$

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(c) The shortfall probability can be evaluated using the formula \( \frac{x - a}{b - a} \) for the distribution function of the uniform distribution:

\[
P(R < 100,000) = \frac{100,000 - (-200,000)}{300,000 - (-200,000)} = \frac{300,000}{500,000} = 0.6
\]

Alternatively, you can evaluate the integral:

\[
\int_{-200,000}^{100,000} f(x) \, dx
\]

(d) Value at Risk
We need to find the (lower) 5% percentile of the distribution of value of \( R \). We can do this using the same formula we used in part (c):

\[
\frac{x - a}{b - a} = 0.05
\]

ie

\[
\frac{x - (-200,000)}{500,000} = 0.05 \quad \Rightarrow \quad x = 0.05 \times 500,000 - 200,000 = -175,000
\]

Therefore, the Value at Risk at the 5% level is \(-(-175,000) = 175,000\).
Why diversification reduces or Eliminates Firm-Specific risk

- Diversification is the process of holding many investments in a portfolio, either across the same asset class (e.g. stocks).
- Risk that affect one of a few firms i.e firm specific risk, can be reduced or even eliminated by investors as they hold more diverse portfolio due to two reasons.
  - Each investment in a diversified portfolio is a much smaller percentage of that portfolio. Thus, any risk that increases or reduces the value of only that investment or a small group of investments will have only a small impact on the overall portfolio.
  - The effect of firm-specific actions on the prices of individual assets in a portfolio can be either positive or negative for each asset for any period. Thus, in large portfolios, it can be reasonably argued that this risk will average out to be zero and thus not impact the overall value of the portfolio.

- In contrast, risk that affects most of all assets in the market will continue to persist even in large and diversified portfolio. For instance, other things being equal, an increase in interest rates will lower the values of most assets in a portfolio.
Mean-Variance portfolio theory

- Mean-variance portfolio theory (MPT—also called modern portfolio theory) assumes that investment decisions are based solely upon risk and return—more specifically the mean and variance of investment return—and that investors are willing to accept higher risk in exchange for higher expected return.

- This can be consistent with the maximisation of expected utility discussed in last section, if the investor is assumed to have a utility function that only uses mean and variance of investment returns, such as the quadratic utility function. It can also be consistent if the distribution of investment returns is a function only of its mean and variance.

- Based upon these and other assumptions, MPT specifies a method for an investor to construct a portfolio that gives the maximum return for a specified risk (variance), or the minimum risk for a specified return, such portfolio are described as efficient.

- A rational investor who prefers more to less and is risk-averse will always choose an efficient portfolio.
If the investor’s utility is known, the MPT allows the investor to choose the portfolio that has the optimal balance between return and risk, as measured by the variance of return, and consequently maximises the investor’s expected utility.

The application of the mean-variance framework to portfolio selection falls conceptually into two parts:

- First the definition of the properties of the portfolios available to the investor— the opportunity set. Here we are looking at the risk and return of the possible portfolios available.
- Second, the determination of how the investor chooses one out of all the feasible portfolios in the opportunity set, i.e. the determination of the investor’s optimal portfolio from those available.
**Definition**

The efficient set is the set of mean-variance choices from the investment opportunity set where for a given variance (or standard deviation) no other investment opportunity offers a higher mean return.

**Remark**

Within the context of mean-variance portfolio theory, risk is defined very specifically as the variance- or equivalently standard deviation- of investment returns.
Assumptions of mean-variance portfolio theory

The application of the mean-variance portfolio theory is based on some important assumptions:

- All expected returns, variances and covariances of pairs of assets are known
- Investors make their decisions purely on the basis of expected return and variance
- Investors are non-satiated (prefers portfolio with higher returns)
- Investors are risk-averse
- There is a fixed single-step time period
- There are no tax or transaction costs
- Assets may be held in any amounts i.e short-selling is possible, we can have infinitely divisible holdings and there are no maximum investment limits.
In specifying the opportunity set it is necessary to make some assumptions about how investors make decisions. Then the properties of portfolios can be specified in terms of relevant characteristics. It is assumed that investors select their portfolios on the basis of:

- The expected return and
- The variance of that return over a single time horizon.

Thus all relevant properties of a portfolio can be specified with just two numbers— the mean return and the variance of the return. The variance (or standard deviation) is known as the risk of the portfolio.

To calculate the mean and variance of return for a portfolio it is necessary to know the expected return on each individual security and also the variance/covariance matrix for the available universe of securities.
The variance/covariance matrix shows the covariance between each pair of the variables. So, if there are three variables 1, 2 and 3 say, then the matrix has the form:

\[
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\]

Where \( \sigma_{ij} \) is the covariance between variables \( i \) and \( j \).

- \( \sigma_{ij} = \sigma_{ji} \) and so the matrix is symmetric about the leading diagonal.
- \( \sigma_{ii} \) is the variance of variable \( i \).

This means that with \( N \) different securities an investor must specify:

- \( N \) expected returns
- \( N \) variance of return
- \( \frac{N(N-1)}{2} \) covariances.

This requirement for an investor to make thousands of estimates of covariances is potentially a major limitation of mean-variance portfolio theory in its general form.
Questions
If you assume that there are 350 shares in an equity index (as there are in the FTSE 350), how many items of data need to be specified for an investor to apply MPT?

Solution
The required number of items of data is:
$$350 + 350 + \frac{350 \times 349}{2} = 61,775$$
Note that this ignores all the other available investments that are not included in the FTSE 350 Index eg non-UK equities, property, bonds etc.
Efficient portfolios

Two further assumptions about investor behavior allow the definition of efficient portfolios:

Assumption

1. Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.
2. Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance.

Definition

A portfolio is efficient if the investor cannot find a better one in the sense that it has either a higher expected return and the same (or lower) variance or lower variance and the same (or higher) expected return, i.e., an efficient portfolio is one that isn’t inefficient.
Suppose an investor can invest in any of the \( N \) securities. A proportion \( w_i \) is invested in security \( S_i \), \( i = 1, \ldots, N \).

Note that

- \( w_i \) is a proportion of the total sum to be invested
- given infinite divisibility, \( w_i \) can assume any value along the real line, subject to the restriction that \( \sum w_i = 1 \)

The return on the portfolio \( R_p \) is: \( R_p = \sum_i w_i R_i \)

where \( R_i \) is the return on security \( S_i \), i.e., the portfolio return is a weighted average of the individual security returns.

The expected return on the portfolio is \( E = E[R_p] = \sum_i w_i E_i \) where \( E_i \) is the expected return on security \( S_i \).

The variance is \( V = \text{var}[R_p] = \sum_i \sum_j w_i w_j \sigma_{ij} \) where \( \sigma_{ij} \) is the covariance of the return on securities \( S_i \) and \( S_j \) and we write \( \sigma_{ii} = V_i \)

So, the lower the covariance between security returns, the lower the overall variance of the portfolio. This means that the variance of a portfolio can be reduced, by investing in securities whose returns are uncorrelated i.e. by diversification.
The case of two securities

- If there are just two securities, $S_A$ and $S_B$, the above expressions reduces to: $E = w_A E_A + w_B E_B$ and $V = w_A^2 + w_B^2 + 2w_A \sigma_{AB}$

- As the proportion invested in $S_A$ is varied a curve is traced in $E - V$ space. The minimum variance can easily be shown to occur when:
  \[
  w_A = \frac{V_B - \sigma_{AB}}{V_A - 2\sigma_{AB} + V_B}.
  \]

- When there are $N$ securities the aim is to choose $w_i$ to minimise $V$ (risk) subject to the constraints: $\sum_i w_i = 1$ and $E = E_p$.

- An alternative approach would be to maximise $E$ subject to:
  $\sum_i w_i = 1$ and $V = V_p$. However, the first approach is usually easier.

- Note carefully that $E$ and $V$ without the subscripts are the portfolio expected return or variance, ie the quantities that we are optimising and that $E_p$ and $V_p$ are the specified values used in the constraints.
Solving minimisation problem

- We wish to:
  - minimise the portfolio variance, $V$
  - subject to the two constraints $\sum_i w_i = 1$ and $E = E_p$
  - by choice of the securities $w_i$, $i = 1, \cdots, N$

- The Lagrangian function is:
  \[ L = V - \lambda (E - E_p) - \mu (\sum_i w_i - 1) \]
  or
  \[ L = \sum_i \sum_j \sigma_{ij} w_i w_j - \lambda (\sum_i E_i w_i - E_p) - \mu (\sum_i w_i - 1) \]

- where:
  - $V$, $E$ and $w_i$ are defined as above
  - $E_p$ and 1 are the constraining constants and
  - $\lambda$ and $\mu$ are known as the Lagrangian multipliers. Remember that we are trying to minimise the variance $V$, subject to the expected return and ”all money invested” constraints.
To find the minimum we set the partial derivatives of $L$ with respect to all the $\sigma_i$ and $\lambda$ and $\mu$ equal to zero. The result is a set of line equations that can be solved.

The partial derivative of $L$ with respect to $x_i$ is:

$$\frac{\partial L}{\partial w_i} = 2 \sum_i w_j \sigma_{ij} - \lambda E_i - \mu$$

The partial derivatives of $W$ with respect to $\lambda$ is:

$$\frac{\partial W}{\partial \lambda} = -\left(\sum_i E_i w_i - E_p\right)$$

and with respect to $\mu$ is:

$$\frac{\partial W}{\partial \mu} = -\left(\sum_i w_i - 1\right)$$

Setting each of these to zero gives:

$$2 \sum_i \sigma_{ij} w_j - \lambda E_i - \mu = 0 \quad \text{(one equation for each of } N \text{ securities)}$$

$$\sum_i w_i E_i = E_p$$

$$\sum_i w_i = 1$$

These equations in can be solved to find the optimal values of the security proportions. These functions can then be substituted into the expression for the portfolio variance, the resulting expression for the portfolio variance as a function of the portfolio expectation being the equation of the minimum variance curve. It is the top half of this curve, ie above the point of global minimum variance, that is the efficient frontier.
Matrix notation

- These $N + 2$ equations are best represented using matrix notation as:

$$Ay = b \text{ where } A = \begin{pmatrix} 2\sigma & -E & -I \\ ET & 0 & 0 \\ IT & 0 & 0 \end{pmatrix}$$

- For example, in the 2-security case:

$$y^T = \{w^T \lambda \mu\}$$

$$b^T = \{00 \cdots E_p1\} \text{ (N zeros)}$$

$$w^T = \{w_1w_2 \cdots w_n\}$$

$$E^T = \{E_1E_2 \cdots E_N\}$$

$$I^T = \{11 \cdots 1\} \text{ (N ones)}$$

- ’T’ denotes transpose i.e although written as row vector, it is actually a column vector
The solution is then: \( y = A^{-1}b \)

By substituting for \( w \) in the equation for \( V = w^T \sigma w \), we can see that the corresponding \( V_p \) is quadratic in \( E_p \) ie the minimum portfolio variance is a quadratic function of the portfolio expected return. This follows immediately from the result that \( w_i \) is linear in \( E_p \). Hence, if \( V_p \) is quadratic in \( w \), then it must be quadratic in \( E_p \).

We now generalise to any \( E \) and \( V \) rather than the specific values of \( E_p \) and \( V_p \). In other words we now look at \((E, V)\) as \( E \) is allowed to vary.

The solution to the problem shows that the minimum variance \( V \) is a quadratic in \( E \) and each \( w_i \) is linear in \( E \).

The usual way of representing the results of the above calculations is by plotting the minimum standard deviation for each value of \( E_p \) as a curve in expected return-standard deviation \((E - \sigma)\) space.
Choosing an efficient portfolio

- We recall that indifference curves join points of equal expected utility in expected return-standard deviation space, ie portfolios that an individual is indifferent between. Note that it is expected utility because we are considering situations involving uncertainty.

- By combining the investor’s indifference curves with the efficient frontier of portfolios, we can determine the investor’s optimal portfolio, ie the portfolio that maximises the investor’s expected utility.

- Utility is maximised by choosing the portfolio on the efficient frontier at the point where the frontier is at tangent to an indifference curve.
Optimal portfolios are also produced for any utility function if investment returns are assumed to be normally distributed. This is very important because investment returns are often modelled using a normal distribution.

If it is felt that the assumptions leading to a two-dimensional mean-variance type portfolio selection model are inappropriate, it is possible to construct models with higher dimensions. For example, skewness could be used in addition to expected return and dispersion measure. It would then be necessary to consider an efficient surface in three dimensions rather than an efficient frontier in two. Clearly, the technique can be extended to more than three dimensions.

For quadratic utility functions the process described above produces optimal portfolios whatever the distribution of returns, because expected utility is uniquely determined if we know the mean and variance of the distribution.
Recall, if the investor has a quadratic utility function, their attitude towards risk can be fully characterised by just the mean and the variance of return. Hence, when maximising expected utility by the choice of portfolio, the investor is concerned only with the first two moments of the investment returns yielded by the investor is concerned only with first two moments of the investment returns yielded by the available portfolios and ignores all other factors.

Questions

1. Explain why the optimal portfolio on the efficient frontier is at the point where the frontier at a tangent to an indifference curve.

2. Why do the investor’s indifference curve slope upward? What determines their gradient?
The optimal portfolio occurs at the point where the indifference curve is tangential to the efficient frontier for the following reasons.

1. The indifference curves that correspond to higher level of expected utility are unattainable as they lie strictly above the efficient frontier.
2. Conversely, lower indifference curves that cut the efficient frontier are attainable, but correspond to a lower level of expected utility.

The highest attainable indifference curve, and corresponding level of expected utility, is therefore the one that is tangential to the efficient frontier. The optimal portfolio occurs at the tangency point, which is in fact the only attainable point of this indifference curve, which is why it is optimal.

The investor’s indifference curves slope upwards because the investor is assumed to be risk-averse and prefer more to less. Consequently, additional expected return yields extra utility, whereas additional risk reduces utility. Thus, any increase (decrease) in risk/standard deviation must be offset by an increase (decrease) in expected return in order to maintain a constant level of expected utility.
The gradient of the indifference curves is determined by the degree of the investor’s risk aversion. The more risk-averse the investor, the steeper the indifference curves—as the investor will require a greater increase in expected return in order to offset any extra risk.
Two- fund separation

Theorem

Two-Fund separation. Each investor will have a utility-maximizing portfolio that is a combination of the risk-free asset and a portfolio (or fund) of risky asset that is determined by the line drawn from the risk-free rate of return tangent to the investor’s efficient set of risky assets.

The straight line (Capital market line) will be the efficient set for all investors and it represents a linear relationship between portfolio risk and return.

Theorem

Capital market Line (CML). If investors have homogeneous belief (ie identical), then they all have the same linear efficient set called the Capital Market Line
Theorem

The single-price law of securities. All securities or combinations of securities that have the same joint distributions of return will have the same price in equilibrium.
While most risk and return models in use in corporate finance agree on the first two step of this process i.e that risk comes from the distribution of actual returns around the expected return and that risk should be measured from the perspective of a marginal investor who is well diversified, they part ways on how to measure the non-diversifiable or market risk.

In this section, we will discuss how the capital asset pricing model (CAPM) approaches the issue of measuring market risk.
Capital Asset Pricing Model was developed almost simultaneously by Sharpe[1963, 1964] and Treynor[1961], and then further develop by Mossin [1966], Linner[1965, 1969] and Black[1972].

- Capital Asset Pricing Model shows that the equilibrium rates of return on all risky assets are a function of their covariance with the market portfolio.
- We recall that the market portfolio is the portfolio held in different quantities by all investors that consist of all risky asset in proportion to their market capitalisation. The proportion of a particular investor’s portfolio consisting of the market portfolio will be determined by their risk-return preference.

The capital asset pricing model tells us about the relationship between risk and return in the security market as a whole, assuming that investors acts in accordance with mean-variance portfolio theory and that the market is in equilibrium.
CAPM assumptions

Capital Asset Pricing Model is developed under the following assumptions about investors and the opportunity set:

1. Investors are risk-averse individuals who maximize the expected utility of their wealth.

2. Investors are price takers and have homogeneous expectations about asset returns that have a joint normal distribution.

3. There exists a risk-free asset such that investors may borrow or lend unlimited amounts at a risk-free rate.

4. The quantities of assets are fixed. Also, all assets are marketable and perfectly divisible.

5. Asset markets are frictionless, the borrowing rate equals the lending rate market line, information is costless and simultaneously available to all investors and no investor believes that they can affect the price of a security by their own actions.

6. All investors have the same one-period horizon.
(7) There are no market imperfections such as taxes, regulations, or restrictions on short selling.

(8) Investors have the same estimates of the expected returns, standard deviations and covariances of securities over the one-period horizon.
Remark

The following are some implications of the above assumptions

- If markets are frictionless, the borrowing rate equals the lending rate, and we are able to develop a linear efficient set called the capital market line,

- If all assets are divisible and marketable, we exclude the possibility of human capital as we usually think of it. In other words, slavery is allowed in the model. We are all able to sell (not rent for wages) various portions of our human capital (e.g., typing ability or reading ability) to other investors at market prices
Remark

- The assumption that investors have homogeneous beliefs means they make decisions based on an identical opportunity set. In other words, no one can be fooled because everyone has the same information at the same time.

  - This will be useful when proving the CAPM where it is required that in equilibrium the market portfolio must be an efficient portfolio. One way to establish its efficiency is to argue that because investors have homogeneous expectations, they will all perceive the same minimum variance opportunity set. Even without a risk-free asset, they will all select efficient portfolios regardless of their individual risk tolerances. We now note that efficiency of the market portfolio and the capital asset pricing model are inseparable, joint hypotheses.

- The model does not require that investors all have the same attitude to risk, only that their views of the available securities are the same and hence that the opportunity set is identical for all investors.

- Also, since all investors maximize the expected utility of their end-of-period wealth, the model is implicitly a one-period model.
Measuring the Market risk of an individual asset

- The risk of any asset to an investor is the risk added on by that asset to the investor’s overall portfolio.
- In CAPM world, where all investors hold the market portfolio, the risk of an individual asset to an investor will be the risk that this asset adds on to the market portfolio.
- Assets that move more with the market portfolio will tend to be riskier than assets that move less, since the movements that are unrelated to the market portfolio will not affect the overall value of the portfolio when an asset is added on to the portfolio.
- Statistically, this added risk is measured by the covariance of the asset with the market portfolio.
- The covariance is a non-standardized measure of market risk to standardize the risk measure, we divide the covariance of each asset with the market portfolio by the variance of the market portfolio.
Calculating Beta

- This yields the beta of the asset:

\[
\text{Beta of an asset } i = \frac{\text{Covariance of asset } i \text{ with Market Portfolio}}{\text{Variance of the Market Portfolio}}
\]

- The beta of any investment in the CAPM is a standardized measure of the risk that it adds to the market portfolio.

- Since the covariance of the market portfolio with itself is its variance, the beta of the market portfolio, and the average asset in it, is one.

- Assets that are riskier than average will have betas that exceed one, while riskless asset will have a beta of zero.
### CAPM: Getting Expected Returns

Expected Return on asset $i = r + \beta_i [E(R_m) - r]$

- = Risk-free rate
- + Beta of asset $i$
- $\star$ (Risk premium on market portfolio)

- The risk premium is the premium demanded by investors for investing in the market portfolio, which includes all assets in the market, instead of investing in a riskless asset.
- The only reason two investments have different expected returns in the capital asset pricing model is because they have different betas.
- The factor $(E_M - r)/\sigma_M$ is often called the market price of risk, where $E_M = E(R_m)$. 

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Questions

Consider Security $A$, which has a standard deviation of investment returns of 4%. If:

- the standard deviation of the market return is 5%
- the correlation between $A$’s return and that of the market is 0.75
- the risk-free rate is 5%
- and the expected return on the market is 10%

then calculate:

(i) the beta of Security $A$
(ii) Security $A$’s expected return.
Solution

(i) Beta of Security is given by:

\[
\beta_A = \frac{\text{Cov}(R_A, R_M)}{\sigma_M^2} = \frac{\rho_{AM}\sigma_A\sigma_M}{\sigma_M^2}
\]

\[
= \frac{0.75 \times 0.004 \times 0.05}{0.05^2} = 0.6
\]

(ii) Expected return of security A is given by

\[
E_A = r + \beta_A (E_M - r)
\]

\[
= 0.05 + 0.6(0.10 - 0.05) = 0.08 \text{ ie } 8\%
\]
Questions

1. You believe that the Beta Alpha Watch Company will be worth $100 per share one year from now. How much are you willing to pay for one share today if the risk-free rate is 8%, the expected rate of return on the market is 18% and the company’s beta is 2.0?

2. The market price of a security is $40, the security’s expected rate of return is 13%, the riskless rate of interest is 7%, and the market risk premium, \([E(R_m) - r]\), is 8%. What will be the security’s current price if its expected future payoff remains the same but the covariance of its rate of return with the market portfolio doubles?
Reference

1. CT 8 study notes, Chapter 1-6.